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Exploring Natural Mates in Euclidean and Minkowski Space

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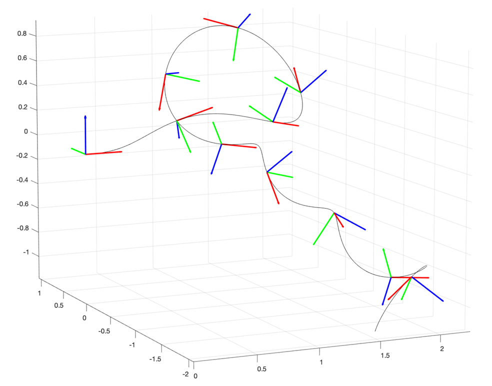
Unit Speed Curves in $\mathbb{R}^3$

Let $\alpha : I \to \mathbb{R}^3$ be a continuously differentiable function parameterized by arc length s. Since $||\alpha'(s)|| = 1$, it is a unit speed curve. We define the Frenet-Serret apparatus as the set $\{\kappa(s), \tau(s), T(s), N(s), B(s)\}$, where $T(s) = \alpha'(s)$ is the unit tangent vector, $\kappa(s) = ||T'(s)||$ is the curvature, $N(s) = \frac{T'(s)}{\kappa(s)}$ is the unit normal vector, $B(s) = T(s) \times N(s)$ is the unit binormal vector, and $\tau(s) = -\langle B'(s), N(s)\rangle$ is the torsion or second curvature. We call the set $\{T(s), N(s), B(s)\}$ the Frenet frame of the curve.

The Frenet-Serret Theorem[4] states that the derivatives of the Frenet frame can be expressed as:

$$\begin{bmatrix} T'(s) \\ N'(s) \\ B'(s) \end{bmatrix} = \begin{bmatrix} 0 & \kappa(s) & 0 \\ -\kappa(s) & 0 & \tau(s) \\ 0 & -\tau(s) & 0 \end{bmatrix} \begin{bmatrix} T(s) \\ N(s) \\ B(s) \end{bmatrix}$$

The Fundamental Theorem of Curves[4] states that any regular curve with $\kappa > 0$ is uniquely determined up to orientation and position by its curvature and torsion. From now on we will only consider curves such that $\kappa > 0$.



(Figure 1) A unit-speed curve and its Frenet frame

Natural Mates

Given some unit-speed curve α with Frenet-Serret apparatus $\{\kappa(s), \tau(s), T(s), N(s), B(s)\}$, we define the unit-speed curve β with Frenet-Serret apparatus $\{\bar{\kappa}(s), \bar{\tau}(s), \bar{T}(s), \bar{N}(s), \bar{B}(s)\}$ such that $\bar{T}(s) = N(s)$. We call β the natural mate of α.

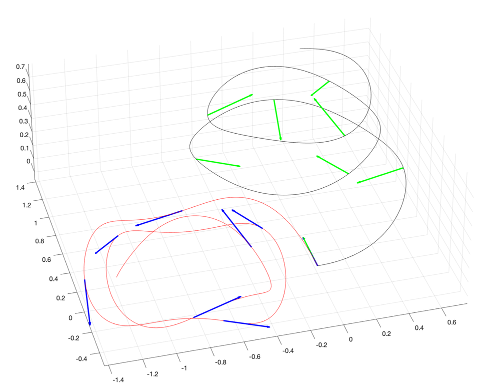
We define[2]:

$$\omega = \sqrt{\tau^2 + \kappa^2} \qquad \sigma = \frac{\kappa^2 (\frac{\tau}{\kappa})'}{(\tau^2 + \kappa^2)^{\frac{3}{2}}}$$

$$\delta = \tau T + \kappa B \qquad \delta^* = -\kappa T + \tau B$$

We can show that the Frenet-Serret apparatus of β in terms of the Frenet-Serret apparatus of α is given by[2]:

$$\bar{\kappa} = \omega \qquad \bar{\tau} = \sigma\omega \qquad \bar{T} = N \qquad \bar{N} = \frac{\delta^*}{\omega} \qquad \bar{B} = \frac{\delta}{\omega}$$



(Figure 2) Curve α in black with normal vectors in green, and α's natural mate β in red with tangent vectors in blue. Note how corresponding normal and tangent vectors along the curves are equal.

2nd Natural Mates

Given some unit speed curve curve α with natural mate β, we say that a curve γ is the second natural mate of α if γ is the natural mate of β. We find that the Frenet-Serret apparatus of γ, $\{\widetilde{\kappa}(s), \widetilde{\tau}(s), \widetilde{T}(s), \widetilde{N}(s), \widetilde{B}(s)\}$, can be expressed in terms of the Frenet-Serret apparatus of α as:

$$\widetilde{\kappa} = \omega\sqrt{\sigma^2 + 1}, \qquad \widetilde{\tau} = \frac{\sigma'}{\sigma^2 + 1},$$

$$\widetilde{T} = \frac{\delta^*}{\omega}, \qquad \widetilde{N} = \frac{-N\omega + \sigma\delta}{\omega\sqrt{\sigma^2 + 1}}, \qquad \widetilde{B} = \frac{\omega\sigma N + \delta}{\omega\sqrt{\sigma^2 + 1}}.$$

From these, we showed that the first and second natural mates must satisfy:

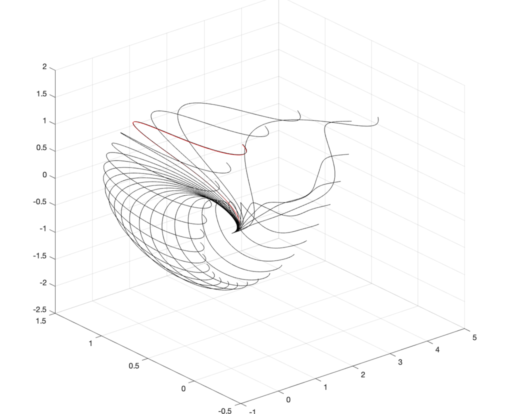
$$\frac{\widetilde{\kappa}'}{\widetilde{\kappa}} = \frac{\bar{\kappa}'}{\bar{\kappa}} + \sigma\widetilde{\tau}$$

Non-Uniqueness of Primitive Curves and Congruence between Natural Mates

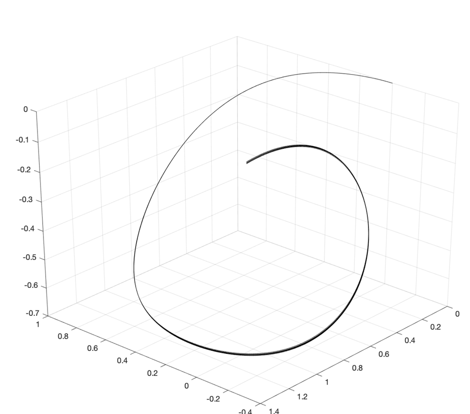
Suppose β is the natural mate of α, does β uniquely determine α? We call α a primitive curve of β. We find that the primitive curve is not uniquely determined by its natural mate, as the curvature and torsion of the primitive curve, κ, τ, given those of the natural mate $\bar{\kappa}, \bar{\tau}$, can only be determined to obey the coupled first order differential equations:

$$\kappa' = \kappa\frac{\bar{\kappa}'}{\bar{\kappa}} - \tau\bar{\tau} \qquad \tau' = \tau\frac{\bar{\kappa}'}{\bar{\kappa}} - \kappa\bar{\tau}$$

Which implies that there is a family of curves whom all have the same natural mate, where each obeys the above differential equations with different initial values of κ and τ.



(Figure 3) Collection of Primitive Curves generated from the red curve



(Figure 4) Collection of Natural Mates of the curves in Figure 3

Another natural question to consider is what if one of the natural mates of some curve is congruent to the original curve, that is, it is the same up to isometry (position and orientation), and by Fundamental Theorem of Curves have the same first and second curvatures. We denote the nth natural mate of α as α_n. We showed that the following statements are equivalent:

- α is planar.
- There exists some $n \in \mathbb{N}$ such that $\alpha \cong \alpha_n$
- $\alpha \cong \alpha_n$ for all $n \in \mathbb{N}$

Thus, if some curve is congruent to any of its natural mates, then it is congruent to all of them, and the curve (and all of its natural mates) must also be planar.

Extending to Minkowski 4-space

Minkowski space is distinguished from our standard Euclidean space in that the way distances is measured is different. While in a 4 dimensional Euclidean space we measure distances using the standard Euclidean metric (inner product)

$$\langle \cdot, \cdot \rangle = dx_0^2 + dx_1^2 + dx_2^2 + dx_3^2.$$

In Minkowski space, one of the dimensions is designated as time-like, and contributes towards the inner product with a negative sign, so the Minkowski metric[3] is expressed as

$$\langle \cdot, \cdot \rangle = -dx_0^2 + dx_1^2 + dx_2^2 + dx_3^2.$$

The other dimensions are referred to as space-like. If the squared length of a vector is negative, we say that the vector is time-like, and if it is positive, we say it is space-like.

Because we are adding another dimension to our space, we need to add another unit vector and curvature function to our Frenet-Serret apparatus, and additionally, since we are now working in Minkowski space, we need to keep track of which of the vectors in the Frenet frame are space-like or time-like. We therefore define our Frenet-Serret apparatus of our unit-speed curve α to be the set $\{\kappa_1(s), \kappa_2(s), \kappa_3(s), A_0(s), A_1(s), A_2(s), A_3(s)\}$, where $\{A_0, A_1, A_2, A_3\}$ is our Frenet Frame, and $\{\kappa_1, \kappa_2, \kappa_3\}$ are our curvatures, which satisfy[1,5]:

$$\begin{aligned} \alpha' &= A_0 & \langle A_0, A_0\rangle &= \epsilon_0 = \pm 1 \\ A_0' &= \kappa_1 A_1 & \langle A_1, A_1\rangle &= \epsilon_1 = \pm 1 \\ A_1' &= \kappa_2 A_2 - \epsilon_0\epsilon_1\kappa_1 A_0 & \langle A_2, A_2\rangle &= \epsilon_2 = \pm 1 \\ A_2' &= \kappa_3 A_3 - \epsilon_1\epsilon_2\kappa_2 A_1 & \langle A_3, A_3\rangle &= \epsilon_3 = \mp 1 \\ A_3' &= -\epsilon_2\epsilon_3\kappa_3 A_2 \\ A_3 &= \epsilon_1 A_0 \wedge A_1 \wedge A_2, \end{aligned}$$

where $\kappa_1, \kappa_2 > 0$. We call ϵ_i the indicator of A_i, which tells us whether A_i is time-like or space-like, where exactly one of $\{\epsilon_0, \epsilon_1, \epsilon_2, \epsilon_3\}$ is -1 for a given curve.

Given these, we define the natural mate the same way as in the Euclidean space. Let β be a unit speed curve with Frenet-Serret apparatus $\{\mu_1(s), \mu_2(s), \mu_3(s), B_0(s), B_1(s), B_2(s), B_3(s)\}$, where $\{B_0, B_1, B_2, B_3\}$ is our Frenet Frame with indicators $\{\eta_0, \eta_1, \eta_2, \eta_3\}$ respectively, and $\{\mu_1, \mu_2, \mu_3\}$ are our curvatures. We say β is the natural mate of α if $B_0 = A_1$.

So far, we have determined that given a curve α, the natural mate β's Frenet-Serret apparatus must satisfy:

$$\begin{aligned} \eta_1\mu_1^2 &= \epsilon_0\kappa_1^2 + \epsilon_2\kappa_2^2 \\ \eta_2\mu_2^2 &= \epsilon_0\epsilon_2\eta_1\frac{\kappa_1^4}{\mu_1^4}\left(\frac{\kappa_2}{\kappa_1}\right)'^2 - \epsilon_0\epsilon_1\epsilon_2\frac{\kappa_2^2\kappa_3^2}{\mu_1^2} \\ B_0 &= A_1 \\ B_1 &= \frac{1}{\mu_1}(\kappa_2 A_2 - \epsilon_0\epsilon_1\kappa_1 A_0) \end{aligned}$$

References

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